

CONSTRUCTION OF AN ORDINARY DIRICHLET SERIES WITH CONVERGENCE BEYOND THE BOHR STRIP

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ABSTRACT. An ordinary Dirichlet series has three abscissae of interest, describing the maximal regions where the Dirichlet series converges, converges uniformly, and converges absolutely. The paper of Hille and Bohnenblust in 1931, regarding the region on which a Dirichlet series can converge uniformly but not absolutely, has prompted much investigation into this region, the “Bohr strip.” However, a related natural question has apparently gone unanswered: For a Dirichlet series with non-trivial Bohr strip, how far beyond the Bohr strip might the series converge? We investigate this question by explicit construction, creating Dirichlet series which converge beyond their Bohr strip.

1. INTRODUCTION

An ordinary Dirichlet series is a function of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

with $s = \sigma + it \in \mathbb{C}$. The region on which a Dirichlet series might be expected to converge is a right half plane, we denote these by

$$\Omega_{\sigma} = \{s \in \mathbb{C} : \Re s > \sigma\}$$

(where \Re denotes the real part) and its closure will be written $\overline{\Omega}_{\sigma}$. To a Dirichlet series we can associate several abscissae:

$$\sigma_a = \inf\{\sigma : \sum a_n n^{-s} \text{ converges absolutely for } s \in \Omega_{\sigma}\}$$

$$\sigma_b = \inf\{\sigma : \sum a_n n^{-s} \text{ converges to a bounded function on } \Omega_{\sigma}\}$$

$$\sigma_c = \inf\{\sigma : \sum a_n n^{-s} \text{ converges for all } s \in \Omega_{\sigma}\}.$$

From the definitions, it is evident that $\sigma_c \leq \sigma_b \leq \sigma_a$. Harald Bohr proved that $\sigma_a - \sigma_b \leq 1/2$ in [2]. In 1931, Hille and Bohnenblust [1] showed that this is sharp; there exist Dirichlet series for which $\sigma_a - \sigma_b = 1/2$, and an explicit construction is provided in [1].

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Let $\Lambda(n)$ be the number of prime factors of $n \in \mathbb{N}$, counted with multiplicity (so $\Lambda(8) = 3$). In [1] they also show that if $\sum a_n n^{-s}$ contains only terms of homogeneity at most M , i.e.

$$\Lambda(n) > M \implies a_n = 0$$

then we have $\sigma_a - \sigma_b \leq \frac{1}{2} - \frac{1}{2M}$, and this is also sharp, which is shown by construction.

Since the publication of [1], there has been much investigation into the gap $\sigma_a - \sigma_b$, and the associated “Bohr strip” $\{s \in \mathbb{C} : \sigma_b < \Re s < \sigma_a\}$ and related issues, we recall some of them here. We would like to mention a survey article in this area by Defant and Schwarting [4].

A key inequality in [1] is the following inequality for M -homogenous polynomials: For each M , there is a constant D_M such that, for an M -homogenous polynomial $\sum_{|\alpha|=M} a_\alpha z^\alpha$ on \mathbb{C}^n we have

$$(1) \quad \left(\sum_{|\alpha|=M} |a_\alpha|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \leq D_M \sup_{z \in \mathbb{D}^n} \left| \sum_{|\alpha|=M} a_\alpha z^\alpha \right|$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and \mathbb{D} is the unit disc. This is a generalization of the Littlewood 4/3 inequality [9], which proves the above result in the case $M = 2$. The original proof in [1] did give a bound on the best possible D_M , but this bound has been substantially improved, see the discussions in [5] and [6] which also contain the most recent improvements to our knowledge.

Another development related to the question of the gap $\sigma_a - \sigma_b$ is the theory of p -Sidon sets (see the discussions in [11], [5]), the inequality (1) shows that the set of monomials $\{z^\alpha : |\alpha| = M\}$ is a $\frac{2M}{M+1}$ -Sidon set, for example.

To produce Dirichlet series with a large gap $\sigma_a - \sigma_b$, in addition to the explicit construction of [1], random methods have been employed, such as in [8] where the existence of ordinary Dirichlet series with $\sigma_a - \sigma_b = 1/2$, (and $\sigma_a - \sigma_b = \frac{1}{2} - \frac{1}{2M}$ for homogeneity M), is shown. Random methods are also employed in sections 4 and 5 of [11], to construct Dirichlet polynomials with small $\|\cdot\|_\infty$ norm and thus obtain bounds on the 1-Sidon constant of the set of “frequencies” $\{\log 1, \dots, \log N\}$ (interpreted as functions on the Bohr compactification of \mathbb{R}).

We mention these developments to note that, for all of this progress, a rather natural question remains: For a Dirichlet series with the “gap” $\sigma_a - \sigma_b$ being large, what can be said about the “gap” $\sigma_b - \sigma_c$ for this same series? To what extent can a Dirichlet series converge beyond its Bohr strip? This is the question we explore here.

We first present a general construction of an ordinary Dirichlet series of homogeneity M . Our technique is based on the method of Walsh matrices used in [10], although we depart from [10] by using non-square matrices. We then prove four bounds on the abscissae of this series, of the following forms:

- $\sigma_b \leq B$
- $\sigma_b \geq 0$
- $\sigma_a \geq A$
- $\sigma_c \leq C$.

The construction only yields a non-trivial result (i.e. $\sigma_a - \sigma_b > 0$, $\sigma_b - \sigma_c > 0$) for the cases $M = 2, 3$. We present the construction for general M nevertheless, because the exposition would not be much clearer for $M = 3$ rather than general M , and because we hope that better bounds might be proved for the general construction which would then yield results beyond $M = 3$. For the cases $M = 2, 3$, we obtain:

$M = 2$. We construct a Dirichlet series of homogeneity $M = 2$ satisfying

$$(2) \quad \sigma_a - \sigma_b = 1/4, \quad \sigma_b - \sigma_c \geq 1/4.$$

Construction of such a Dirichlet series (or even proof of its existence) is, to our knowledge, a new result. Note that $1/4$ is the optimal value of $\sigma_a - \sigma_b$, given that $M = 2$.

$M = 3$. Here, for any value $\rho_1 \in (0, 1)$, we construct a Dirichlet series of homogeneity $M = 3$ which satisfies $\sigma_a - \sigma_c \geq 1/3$, and we furthermore have some specific control over $\sigma_b \in (\sigma_c, \sigma_a)$:

$$\sigma_a - \sigma_b \geq \frac{1 + \rho_1}{6}, \quad \sigma_b - \sigma_c \geq \frac{1 - \rho_1}{9}.$$

For $\rho_1 > 1/2$, we see that the value of $\sigma_a - \sigma_b$ is larger than $1/4$, so this construction does represent a result that cannot be achieved with only terms of homogeneity at most two. If we pick, for example, $\rho_1 = 3/4$, then we have

$$\sigma_a - \sigma_b \geq 7/24, \quad \sigma_b - \sigma_c \geq 1/36.$$

Note that, for $M = 3$, unfortunately the current construction does not produce values for $\sigma_a - \sigma_b$, $\sigma_b - \sigma_c$ that couldn't be replicated by a Dirichlet series with existing constructions. For instance, using the standard Hille-Bohnenblust construction for $M = 3$ and adding a properly shifted Dirichlet series with a given value of $\sigma_a - \sigma_c$ (such as the alternating zeta function) will produce a series that has these properties. This can be done only using terms of homogeneity three as well; simply use a version of the alternating zeta function which contains only terms of homogeneity three (and is "alternating" on these terms), we leave details to the interested reader. However, such a series, being more simply constructed, does not afford control on the individual coefficients. To our knowledge, ours is the first construction which exhibits a Dirichlet series having terms of homogeneity exactly three for which it is proved that $\sigma_a - \sigma_b > 1/4$, $\sigma_b - \sigma_c > 0$ and for which we have substantial knowledge regarding the individual coefficients.

Our hope is that the method shown here, since it gives specific control over each abscissa, without any "tricks" of adding another (unrelated) Dirichlet series, could be extended, specifically by improving the estimate in section 6.

In section 2 we present the general construction. In sections 3 and 4, we prove the "easy" bounds: an upper bound on σ_b and a lower bound on σ_a . These first two bounds yield the classic Hille-Bohnenblust-type Dirichlet series, for each homogeneity M . In sections 5, 6 and 7 we prove the "hard" bounds, showing that our Dirichlet series is unbounded on any Ω_σ with $\sigma < 0$, and then showing that our Dirichlet series converges (i.e. the sequence of partial sums converges) at a point $s = -\epsilon$ on the negative real axis. Once all four bounds are proved, in section 8 we derive the results for $M = 2, 3$.

In section 6 we isolate one of the key estimates, a basic size estimate on all partial sums of a certain set of complex numbers of modulus one. It seems some improvement should be possible, given that the arguments are spread over the unit circle.

2. GENERAL CONSTRUCTION

For a polynomial Q in complex variables z_1, z_2, \dots , define the Wiener norm $\|Q\|_W$ to be the sum of the absolute values of the coefficients of Q , and let

$$\|Q\|_\infty = \max\{|Q| : |z_i| \leq 1\}.$$

Let $\|\cdot\|$ denote the usual norm $\|x\|^2 = \sum |x_i|^2$.

The construction here differs from standard constructions of this type, because the matrices we use will not necessarily be square.

For $r \in \mathbb{N}$, let $\omega = \omega_r$ be the primitive r th root of unity $e^{2\pi i/r}$. For $r_1 \leq r_2$, let $B^{(r_2, r_1)} : \mathbb{C}^{r_1} \rightarrow \mathbb{C}^{r_2}$ be the ‘‘Walsh matrix’’ defined by

$$b_{ij} = \omega_{r_2}^{ij}, \quad i = 0, 1, \dots, r_2 - 1, \quad j = 0, 1, \dots, r_1 - 1.$$

So, for $v \in \mathbb{C}^{r_1}$, $B^{(r_2, r_1)}$ satisfies $\|B^{(r_2, r_1)}v\|^2 = r_2\|v\|^2$.

Let $M \in \mathbb{N}$, and suppose $r_1 \leq \dots \leq r_{M+1}$. Suppose we have M sets of complex numbers, with the j th set having r_j elements:

$$z_0^{(1)}, \dots, z_{r_1-1}^{(1)}, z_0^{(2)}, \dots, z_{r_2-1}^{(2)}, \dots, z_0^{(M)}, \dots, z_{r_M-1}^{(M)}.$$

Let $D^{(j)}$ be the $r_j \times r_j$ diagonal matrix with the diagonal entry $d_{ii} = z_i^{(j)}$. We will abbreviate

$$B^{2,1} = B^{(r_2, r_1)}, \quad B^{3,2} = B^{(r_3, r_2)} \quad \text{etc.}$$

Let $u = (1, \dots, 1) \in \mathbb{C}^{r_1}$, and consider the vector

$$B^{M+1, M} D^{(M)} B^{M, M-1} D^{(M-1)} \dots B^{3,2} D^{(2)} B^{2,1} D^{(1)} u \in \mathbb{C}^{r_{M+1}}.$$

Suppose that each $z_i^{(j)}$ satisfies $|z_i^{(j)}| \leq 1$. Then we have

$$\begin{aligned} \|B^{M+1, M} D^{(M)} \dots B^{2,1} D^{(1)} u\|^2 &= r_{M+1} \|D^{(M)} \dots B^{2,1} D^{(1)} u\|^2 \\ &\leq r_{M+1} \|B^{M, M-1} D^{(M-1)} \dots B^{3,2} D^{(2)} B^{2,1} D^{(1)} u\|^2 \\ &= \dots \\ &= r_{M+1} r_M \dots r_2 \|u\|^2 \\ &= r_{M+1} r_M \dots r_2 r_1 \\ &= \Pi_1^{M+1} r_j. \end{aligned}$$

This means that each coordinate of $B^{M+1, M} D^{(M)} \dots B^{2,1} D^{(1)} u$ has size less than or equal to $\left(\Pi_1^{M+1} r_j\right)^{1/2}$. In particular the 0th coordinate, which is

$$\sum_{i_1=0}^{r_1-1} \dots \sum_{i_M=0}^{r_M-1} z_{i_1}^{(1)} z_{i_2}^{(2)} \dots z_{i_M}^{(M)} \omega_{r_2}^{i_1 i_2} \omega_{r_3}^{i_2 i_3} \dots \omega_{r_M}^{i_{M-1} i_M}$$

satisfies

$$\left| \sum_{i_1=0}^{r_1-1} \dots \sum_{i_M=0}^{r_M-1} z_{i_1}^{(1)} \dots z_{i_M}^{(M)} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_M}^{i_{M-1} i_M} \right| \leq \left(\Pi_1^{M+1} r_j\right)^{1/2} \quad \text{when } |z_i^{(j)}| \leq 1.$$

We define

$$(3) \quad Q = Q_{r_1, \dots, r_M} = \sum_{i_1}^{r_1-1} \cdots \sum_{i_M}^{r_M-1} z_{i_1}^{(1)} \cdots z_{i_M}^{(M)} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} .$$

Considering Q as a polynomial in the variables $z_i^{(j)}$, we have

$$(4) \quad \|Q\|_W = \Pi_1^M r_j$$

$$(5) \quad \|Q\|_\infty \leq \left(\Pi_1^{M+1} r_j \right)^{1/2} .$$

Note that Q is not just a polynomial in the $z_i^{(j)}$, but is in fact a multilinear function of the vectors $z^{(1)}, z^{(2)}, \dots, z^{(M)}$. Note that Q depends on r_1, \dots, r_M , so in our case, where the goal is to make the Wiener norm of Q as large as possible, and make the supremum norm as small as possible, we set r_{M+1} to the smallest possible value, namely r_M .

We denote the floor function by $\lfloor x \rfloor = \max\{n \in \mathbb{N} : n \leq x\}$. Next, we define some key parameters of this construction:

$$\begin{aligned} L &\in \mathbb{N} \\ \rho_1 &\leq \dots \leq \rho_{M-1} \in [0, 1] \quad , \quad \rho_M = 1 \\ r_1 &= \lfloor 2^{\rho_1 L} \rfloor, \dots, r_M = \lfloor 2^{\rho_M L} \rfloor \end{aligned}$$

where ρ_1, \dots, ρ_M are fixed and L is an index which will range over \mathbb{N} . Notice that the r_j depend on L , so to be proper we might write $r_j^{(L)}$; we will not do so since the value of L will be clear from context. We define

$$Q^L = Q_{r_1, \dots, r_M} .$$

When one has a polynomial Q in the complex variables z_1, z_2, \dots , we can create a Dirichlet polynomial P via the substitution

$$P(s) = Q(2^{-s}, 3^{-s}, \dots) .$$

We have just defined a family of polynomials $\{Q^L\}$, each being homogenous of degree M . To create the Dirichlet polynomials and Dirichlet series that we want, we will use polynomials Q from this family.

Let p_k be the k th prime number. Note that, by the Prime Number Theorem [7] (in fact, what we need is weaker than the Prime Number Theorem), we have c, C such that

$$ck \log k \leq p_k \leq Ck \log k .$$

This is all we will need about the distribution of primes.

Now, we define a dyadic family of disjoint sets of primes: Let the sets $K_L^{(j)}$ be defined by

$$K_L^{(j)} = \{(M+j-1)2^L + x : x = 0, \dots, r_j - 1\}$$

and then the family of sets of primes is defined by

$$\Pi_L^{(j)} = \{p_k : k \in K_L^{(j)}\} .$$

For convenience, when the value of L is clear from context, we denote the i th element of $K_L^{(j)}$ by

$$k_i^{(j)} = (M + j - 1)2^L + i.$$

Note that all of the $\Pi_L^{(j)}$ are pairwise disjoint.

The terms in the Dirichlet polynomial P_L will involve only those n which are a product of a single element from each $\Pi_L^{(j)}$. Define

$$\Pi_L^\times = \Pi_L^{(1)} \cdot \Pi_L^{(2)} \cdots \Pi_L^{(M)} = \left\{ n = p_{k_{i_1}^{(1)}} p_{k_{i_2}^{(2)}} \cdots p_{k_{i_M}^{(M)}} , \quad i_j \in \{0, \dots, r_j - 1\}, \quad L \in \mathbb{N} \right\}$$

and define

$$P_L(s) = Q^L \left(p_{k_0^{(1)}}^{-s}, \dots, p_{k_{r_1-1}^{(1)}}^{-s}, p_{k_0^{(2)}}^{-s}, \dots, p_{k_{r_2-1}^{(2)}}^{-s}, \dots, p_{k_0^{(M)}}^{-s}, \dots, p_{k_{r_M-1}^{(M)}}^{-s} \right).$$

In other words,

$$P_L = \sum_{n \in \mathbb{N}} \gamma_n n^{-s}$$

where

$$\gamma_n = \begin{cases} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} & \text{if } n \in \Pi_L^\times, \quad n = p_{k_{i_1}^{(1)}} \cdots p_{k_{i_M}^{(M)}} \\ 0 & \text{else} \end{cases}.$$

At this point, the idea is to consider the Dirichlet series $\sum_L c_L P_L$ with some coefficients c_L . However, instead of defining the series in this way, we will define it “directly” by defining its coefficients a_n . This will be convenient since we want to consider the conditional convergence with proper care.

We will let $X > 0$ be a fixed real number which is not yet specified, but X will only depend on M and ρ_1, \dots, ρ_M . So, consider the Dirichlet series

$$(6) \quad f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

where

$$(7) \quad \begin{aligned} \beta_L & \quad \text{is fixed but to-be-determined, with } |\beta_L| = 1 \\ a_n &= \begin{cases} \beta_L 2^{-XL} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} & \text{if } n \in \Pi_L^\times, \quad n = p_{k_{i_1}^{(1)}} \cdots p_{k_{i_M}^{(M)}} \\ 0 & \text{else} \end{cases} \end{aligned}$$

So, the idea is

$$f(s) = \sum_{L=1}^{\infty} \beta_L 2^{-XL} P_L(s)$$

although (6), (7) is the proper definition.

Having this general construction, we will now prove four bounds on the abscissae of f , in the next five sections. We begin with the two easier bounds: an upper bound on σ_b and a lower bound on σ_a .

3. AN UPPER BOUND ON THE ABCISSA OF BOUNDEDNESS

Let $\sigma > 0$, let the real part of s be greater than or equal to σ , and let n_* be the smallest n in Π_L^\times ,

$$n_* = p_{k_0^{(1)}} p_{k_0^{(2)}} \cdots p_{k_0^{(M)}} .$$

We have

$$\begin{aligned} P_L(s) &= \sum_n \gamma_n n^{-s} \\ &= n_*^{-\sigma} \sum_n \gamma_n n^{-s} n_*^\sigma \\ &= n_*^{-\sigma} Q^L \left(p_{k_0^{(1)}}^\sigma p_{k_0^{(1)}}^{-s}, \dots, p_{k_0^{(1)}}^\sigma p_{k_{r_1-1}^{(1)}}^{-s}, \dots, p_{k_0^{(M)}}^\sigma p_{k_0^{(M)}}^{-s}, \dots, p_{k_0^{(M)}}^\sigma p_{k_{r_M-1}^{(M)}}^{-s} \right) \end{aligned}$$

and

$$|p_{k_0^{(j)}}^\sigma p_{k_i^{(j)}}^{-s}| \leq 1 \quad \forall i, j .$$

Recall (5):

$$\|Q^L\|_\infty \leq 2^{(\rho_1 + \dots + \rho_M + 1)L/2} .$$

We also have $n_* \geq c(M2^L)^M \geq c_M 2^{LM}$ by the Prime Number Theorem, so that

$$|P_L(s)| \leq c_M^{-\sigma} 2^{-\sigma LM} 2^{(\rho_1 + \dots + \rho_M + 1)L/2}$$

and therefore

$$\begin{aligned} |\beta_L 2^{-XL} P_L(s)| &\leq c_M^{-\sigma} 2^{-XL} 2^{-\sigma LM} 2^{(\rho_1 + \dots + \rho_M + 1)L/2} \\ &= c_M^{-\sigma} 2^{[(1/2)(\rho_1 + \dots + \rho_M + 1) - \sigma M - X]L} . \end{aligned}$$

We see that, if $(1/2)(\rho_1 + \dots + \rho_M + 1) - \sigma M - X < 0$, then $\sum_L \beta_L 2^{-XL} P_L$ defines a bounded holomorphic function in the half plane Ω_σ .

By inspection in some possibly distant right half plane we can conclude that

$$f = \sum_L \beta_L 2^{-XL} P_L$$

in this distant right half plane (since the two Dirichlet series will converge absolutely). So, $\sum_L \beta_L 2^{-XL} P_L$ gives an analytic continuation of f to a bounded function on Ω_σ , and therefore by a classic theorem of Bohr [3] we know that the Dirichlet series for f converges on Ω_σ , and f is bounded there, so $\sigma_b \leq \sigma$.

We have shown that if $\sigma > 0$ and $(1/2)(\rho_1 + \dots + \rho_M + 1) - \sigma M - X < 0$ then $\sigma_b \leq \sigma$. So, if

$$X \leq (1/2)(\rho_1 + \dots + \rho_M + 1)$$

and we choose any σ satisfying

$$\sigma > (1/2M)(\rho_1 + \dots + \rho_M + 1) - X/M$$

then $\sigma_b \leq \sigma$, and therefore by taking the infimum over σ we have proved:

Proposition 1. *Let f be the Dirichlet series defined by (6) and (7). If*

$$X \leq (1/2)(\rho_1 + \cdots + \rho_M + 1)$$

then we have

$$\sigma_b \leq (1/2M)(\rho_1 + \cdots + \rho_M + 1) - X/M.$$

4. ABSCISSA OF ABSOLUTE CONVERGENCE

We prove a lower bound on σ_a . Note that, by the Prime Number Theorem, we have c, C such that

$$ck \log k \leq p_k \leq Ck \log k$$

and so we have

$$\max\{n : n \in \Pi_L^\times\} \leq \left(\max\{p_k : k \in K_L^{(j)}\} \right)^M.$$

We recall that all $k \in K_L^{(j)}$ satisfy $k \leq M2^{L+1}$, and so

$$p_k \leq M2^{L+2}(\log M + (L+1)\log 2).$$

Therefore, with C_M being a constant depending on M ,

$$\max\{p_k : k \in K_L^{(j)}\} \leq C_M 2^L L$$

and so

$$(8) \quad \max\{n : n \in \Pi_L^\times\} \leq C_M^M 2^{ML} L^M.$$

Also, we have

$$|\Pi_L^\times| = r_1 \cdots r_M \geq \frac{r_{M+1}^{\rho_1}}{2} \cdots \frac{r_{M+1}^{\rho_M}}{2} \geq 2^{-M} 2^{(\rho_1 + \cdots + \rho_M)L}.$$

Therefore, we calculate:

$$\begin{aligned} \sum |a_n| n^{-\sigma} &= \sum_{L=1}^{\infty} 2^{-XL} \sum_{n \in \Pi_L^\times} n^{-\sigma} \\ &\geq \sum_{L=1}^{\infty} 2^{-XL} |\Pi_L^\times| C_M^{-\sigma} 2^{-\sigma ML} L^{-\sigma M} \\ &\geq C_M^{-\sigma} 2^{-M} \sum_{L=1}^{\infty} 2^{-XL} 2^{(\rho_1 + \cdots + \rho_M)L} 2^{-\sigma ML} L^{-\sigma M} \\ &= C_M^{-\sigma} 2^{-M} \sum_{L=1}^{\infty} 2^{[(\rho_1 + \cdots + \rho_M) - \sigma M - X]L} L^{-\sigma M}. \end{aligned}$$

If $(\rho_1 + \cdots + \rho_M) - \sigma M - X > 0$, i.e. if

$$\sigma < (1/M)(\rho_1 + \cdots + \rho_M) - X/M$$

then the above sum is infinite, so we have

Proposition 2. *Let f be the Dirichlet series defined by (6) and (7). We have*

$$\sigma_a \geq (1/M)(\rho_1 + \cdots + \rho_M) - X/M.$$

At this point, we note that we have produced the classic Hille-Bohnenblust construction. With propositions 1 and 2, we see that as long as we choose

$$X \leq (1/2)(\rho_1 + \dots + \rho_M + 1)$$

then we have

$$\sigma_a - \sigma_b \geq \frac{1}{2M}(\rho_1 + \dots + \rho_M) - \frac{1}{2M}.$$

For any value of M , by choosing $\rho_1 = \dots = \rho_M = 1$ (and $X = 0$ for instance), the Dirichlet series f has terms of homogeneity exactly M and $\sigma_a - \sigma_b \geq \frac{1}{2} - \frac{1}{2M}$, the largest possible gap between σ_a and σ_b .

5. PROVING $\sigma_b \geq 0$

To show that f becomes unbounded if we cross the abscissa $\sigma = 0$, i.e. to prove $\sigma_b \geq 0$, we will demonstrate that (under certain conditions on X) the partial sums of f achieve arbitrarily large values on the imaginary axis. This proves the bound because, if $\sigma_b < 0$ then by classic results [3] the partial sums of f converge uniformly to f on the line $\sigma = 0$. Uniform convergence implies that there is some large N' , such that

$$\forall N \geq N', \left\| \sum_{n=1}^{N'} a_n n^{-it} \right\|_{\infty} \leq 2 \|f\|_{\infty}.$$

In particular, with $\sigma_b < 0$ the partial sums cannot achieve arbitrarily large values. Note that we will not show that f itself is actually unbounded on the imaginary axis (this is fortunately not necessary; it may well not be true).

To prove that the partial sums achieve arbitrarily large values on the imaginary axis, we will show that the first finitely many L of the terms $\beta_L 2^{-XL} P_L$ have very similar arguments at some point on the imaginary axis. Furthermore, we want to prove this while still retaining complete flexibility to choose $\{\beta_L\}$. This can be accomplished because, for any choice of $\{\beta_L\}$, once we consider a particular partial sum, we can search the entire imaginary axis to choose the argument of P_L at will, for each of finitely many L (simultaneously) using Kronecker's theorem.

The plan is as follows:

- (1) Fix a large number K ; the partial sums will then be constructed to exceed K .
- (2) Let $L_K = 2^{M+1}K$; this is chosen so $\sum_{L \leq L_K} 2^{-XL} \|Q^L\|_{\infty} > 2K$ (for the right X).
- (3) For each $L \leq L_K$, find z_L such that $\beta_L Q^L(z_L) = \|Q^L\|_{\infty}$.
- (4) By Kronecker's theorem, find $t = t_K$ such that $\left(p_{k_0^{(1)}}^{-it_K}, \dots, p_{k_{r_M-1}^{(M)}}^{-it_K} \right) \approx z_L$ is satisfied simultaneously for all $L \leq L_K$.

Then, $P_L(it_K) \approx Q^L(z_L)$ and so

$$\begin{aligned} \sum_{L \leq L_K} 2^{-XL} \beta_L P_L(it_K) &\approx \sum_{L \leq L_K} 2^{-XL} \beta_L Q^L(z_L) \\ &= \sum_{L \leq L_K} 2^{-XL} \|Q^L\|_{\infty} \end{aligned}$$

where the first sum is indeed a partial sum of the Dirichlet series for f at the point $s = it_K$. The fact that this is in fact a partial sum of the Dirichlet series for f is not a complete triviality, but it is true; it equals $\sum_{n=1}^N a_n n^{-it_K}$ where $N = \max\{n : n \in \Pi_{L_K}^\times\}$. The important point is that, if $L_1 < L_2$ then

$$\forall n_1 \in \Pi_{L_1}^\times, \forall n_2 \in \Pi_{L_2}^\times, \text{ we have } n_1 < n_2.$$

Now, the details. We first fix a (large) K .

Recall that

$$Q^L = \sum_{i_1}^{r_1-1} \cdots \sum_{i_M}^{r_M-1} z_{i_1}^{(1)} \cdots z_{i_M}^{(M)} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M}.$$

Recall inequality (1) from the introduction: For any M -homogenous polynomial in n variables, we have

$$\left(\sum_{|\alpha|=M} |a_\alpha|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \leq D_M \sup_{z \in \mathbb{D}^n} \left| \sum_{|\alpha|=M} a_\alpha z^\alpha \right|.$$

By the maximum modulus principle, Q^L will attain its supremum on \mathbb{T}^n . So, for each L , let $(z_i^{*(j)}) = (z_i^{*(j)}(L))$ be a point on \mathbb{T}^n where the supremum is attained. We have

$$\begin{aligned} |Q^L((z_i^{*(j)}))| &\geq D_M^{-1} \left(\sum_{|\alpha|=M} |a_\alpha|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} = D_M^{-1} (r_1 \cdots r_M)^{\frac{M+1}{2M}} \\ (9) \qquad \qquad \qquad &\geq 2^{-M} 2^{(\rho_1 + \cdots + \rho_M) \frac{M+1}{2M}} L. \end{aligned}$$

Next, note that since Q^L is in fact multilinear on $\mathbb{C}^{r_1} \times \cdots \times \mathbb{C}^{r_M}$, we can multiply by a constant in the \mathbb{C}^{r_1} coordinate and then factor out that constant by linearity: for any τ_L with $|\tau_L| = 1$, we have

$$Q^L(\tau_L z_0^{*(1)}, \tau_L z_1^{*(1)}, \dots, \tau_L z_{r_1-1}^{*(1)}, z_0^{*(2)}, \dots, z_{r_2-1}^{*(2)}, \dots, z_{r_M-1}^{*(M)}) = \tau_L Q^L((z_i^{*(j)})).$$

Let y_L be the point appearing in the equation above:

$$\begin{aligned} y_{i,j,L} &= \tau_L z_i^{*(j)}(L) & \text{for } j = 1 \\ y_{i,j,L} &= z_i^{*(j)}(L) & \text{for } j > 1 \end{aligned}$$

for all L , and so $Q^L(y_L) = \tau_L Q^L((z_i^{*(j)}))$. We now fix τ_L such that

$$(10) \qquad \beta_L Q^L(y_L) = \beta_L \tau_L Q^L((z_i^{*(j)})) = \left| Q^L((z_i^{*(j)})) \right|.$$

We will now need Kronecker's Theorem [7]:

Theorem 3 (Kronecker's Theorem). *Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be rationally independant. Then, the map*

$$t \longrightarrow (e^{i\lambda_1 t}, \dots, e^{i\lambda_n t})$$

from \mathbb{R} to \mathbb{T}^n has dense range in \mathbb{T}^n .

The set $\{\log p : p \text{ is prime}\}$ is rationally independant, so by Kronecker's Theorem let t_K be chosen so that

$$|p_{k_i^{(j)}}^{-it_K} - y_{i,j,L}| \leq 2^{-(2M+1)L}$$

is satisfied for all i, j and all $L \leq L_K$ (simultaneously). Write this as

$$p_{k_i^{(j)}}^{-it_K} = y_{i,j,L} + \delta_{i,j,L}.$$

So,

$$\begin{aligned} P_L(it_K) &= Q^L(p_1^{-it_K}, p_2^{-it_K}, \dots) \\ &= \sum_{i_1, \dots, i_M} \left(\tau_L z_{i_1}^{*(1)} + \delta_{i_1,1,L} \right) \left(z_{i_2}^{*(2)} + \delta_{i_2,2,L} \right) \cdots \left(z_{i_M}^{*(M)} + \delta_{i_M,M,L} \right) \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \\ &= \sum_{i_1, \dots, i_M} \left(\tau_L z_{i_1}^{*(1)} z_{i_2}^{*(2)} \cdots z_{i_M}^{*(M)} + \sum_{\delta} \right) \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \end{aligned}$$

where \sum_{δ} represents the remaining terms in the binomial expansion; we have

$$\left| \sum_{\delta} \right| \leq 2^M 2^{-(2M+1)L}.$$

Continuing the above equality,

$$\begin{aligned} P_L(it_K) &= \tau_L Q^L \left((z_i^{*(j)}) \right) + \sum_{i_1, \dots, i_M} \left(\sum_{\delta} \right) \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} \\ &= \tau_L Q^L \left((z_i^{*(j)}) \right) + \delta_L. \end{aligned}$$

The last line above serves to define δ_L , and we have

$$|\delta_L| \leq 2^{(\rho_1 + \dots + \rho_M)L} 2^M 2^{-(2M+1)L} \leq 2^{ML} 2^M 2^{-(2M+1)L} \leq 2^{-L}.$$

Let $N_K = \max\{n : n \in \Pi_{L_K}^{\times}\}$. Then

$$\begin{aligned} \sum_{n=1}^{N_K} a_n n^{-it_K} &= \sum_{L=1}^{L_K} \beta_L 2^{-XL} P_L(it_K) \\ &= \sum_{L=1}^{L_K} \beta_L 2^{-XL} \left(\tau_L Q^L \left((z_i^{*(j)}) \right) + \delta_L \right) \\ &= \sum_{L=1}^{L_K} \beta_L 2^{-XL} \tau_L Q^L \left((z_i^{*(j)}) \right) + \sum_{L=1}^{L_K} \beta_L 2^{-XL} \delta_L \\ &= \sum_{L=1}^{L_K} 2^{-XL} \left| Q^L \left((z_i^{*(j)}) \right) \right| + \sum_{L=1}^{L_K} \beta_L 2^{-XL} \delta_L \quad [\text{by (10)}]. \end{aligned}$$

We see that

$$\sum_{L=1}^{L_K} 2^{-XL} \left| Q^L \left((z_i^{*(j)}) \right) \right| \geq \sum_{L=1}^{L_K} 2^{-XL} 2^{-M} 2^{(\rho_1 + \dots + \rho_M) \frac{M+1}{2M} L}.$$

We now can choose the value of X that we would like:

$$X = (\rho_1 + \cdots + \rho_M) \frac{M+1}{2M} .$$

With this,

$$\sum_{L=1}^{L_K} 2^{-XL} \left| Q^L \left((z_i^{*(j)}) \right) \right| \geq 2^{-M} L_K \geq 2K$$

and $\left| \sum_{L=1}^{L_K} \beta_L 2^{-XL} \delta_L \right| \leq \sum 2^{-XL} 2^{-L} \leq 1 \leq K$, so we have

$$\left| \sum_{n=1}^{N_K} a_n n^{-it_K} \right| \geq (2K) - (K) = K .$$

For arbitrary K , we have found a partial sum of f which, at a point on the imaginary axis, assumes a value having modulus larger than K and so we have proved

Proposition 4. *Let f be the Dirichlet series defined by (6) and (7). With*

$$X = (\rho_1 + \cdots + \rho_M) \frac{M+1}{2M}$$

and for any choice of β_L , $|\beta_L| = 1$, we have $\sigma_b \geq 0$.

6. A KEY ESTIMATE

Before presenting the final bound (the upper bound on σ_c) we will require a size estimate on a sum of the following form:

$$\sum_{n \in \Pi_L^\times, n \leq P} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M}$$

for general P . This sum is adding terms on the unit circle with widely varying arguments, so a high degree of cancellation can be hoped for. Unfortunately, no sophisticated or impressive bound has been obtained by the current author; we will simply isolate the i_M index, sum the resulting one-variable geometric series, and then bound by absolute values. Recalling

$$n = p_{k_{i_1}}^{(1)} \cdots p_{k_{i_M}}^{(M)} \longleftrightarrow (i_1, \dots, i_M)$$

the one key observation is that, because we selected the primes in increasing order, the set

$$\{(i_1, \dots, i_M) : n \leq P\}$$

has the following weak convexity property: if we fix (i_1, \dots, i_{M-1}) , then the set

$$\{i_M : (i_1, \dots, i_M) \text{ satisfies } n \leq P\}$$

is an “interval” of natural numbers, meaning that it equals every natural number between some (unspecified) lower and upper bounds, call them l and u respectively. Leaving out

the terms with $i_{M-1} = 0$, we have:

$$\begin{aligned}
& \sum_{n \in \Pi_L^\times, n \leq P, i_{M-1} \neq 0} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_M}^{i_{M-1} i_M} \\
&= \sum_{(i_1, \dots, i_{M-1}), i_{M-1} \neq 0} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_{M-1}}^{i_{M-2} i_{M-1}} \sum_{i_M: (i_1, \dots, i_M) \text{ satisfies } n \leq P} \omega_{r_M}^{i_{M-1} i_M} \\
&= \sum_{(i_1, \dots, i_{M-1}), i_{M-1} \neq 0} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_{M-1}}^{i_{M-2} i_{M-1}} \sum_{i_M=l}^u \omega_{r_M}^{i_{M-1} i_M} \\
&= \sum_{(i_1, \dots, i_{M-1}), i_{M-1} \neq 0} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_{M-1}}^{i_{M-2} i_{M-1}} \left(\frac{\omega_{r_M}^a - \omega_{r_M}^b}{1 - \omega_{r_M}^{i_{M-1}}} \right).
\end{aligned}$$

Noting that on $[-\pi, \pi]$ there is some small c such that $1 - \cos x \geq cx^2$, we have

$$\begin{aligned}
|1 - \omega_{r_M}^{i_{M-1}}|^2 &= 2(1 - \cos(2\pi i_{M-1}/r_M)) \\
&\geq 2c(2\pi i_{M-1}/r_M)^2
\end{aligned}$$

and so there is an absolute constant c' such that $|1 - \omega_{r_M}^{i_{M-1}}| \geq c' i_{M-1}/r_M$.

Now, bounding with absolute values, including those terms with $i_{M-1} = 0$, and estimating $\sum_{i=1}^K i^{-1} \leq C \log(K+1)$, we have

$$\begin{aligned}
\left| \sum_{n \in \Pi_L^\times, n \leq P} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_M}^{i_{M-1} i_M} \right| &\leq c'^{-1} \sum_{(i_1, \dots, i_{M-2})} 2 \sum_{i_{M-1} \geq 1} \frac{r_M}{i_{M-1}} + \sum_{(i_1, \dots, i_M): i_{M-1}=0} 1 \\
&\leq 2c'^{-1} C r_1 \dots r_{M-2} r_M (\log(r_{M-1} + 1) + 1).
\end{aligned}$$

Note that $r_j \leq r_{M+1}^{\rho_j} \leq 2^{\rho_j L}$. With C'_M being another constant depending on M ,

$$(11) \quad \left| \sum_{n \in \Pi_L^\times, n \leq P} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_M}^{i_{M-1} i_M} \right| \leq C'_M 2^{(\rho_1 + \dots + \rho_{M-2} + \rho_M)L} L.$$

7. CONVERGENCE ON THE NEGATIVE REAL AXIS

Note that, after proving the bounds on the abscissae of f in the preceding sections, we still have the freedom to choose β_L . We will now use β_L to arrange a large amount of cancellation in the partial sums of f at a certain point $s = -\epsilon$ (to be determined) on the negative real axis.

Fix some $\epsilon > 0$, and consider a partial sum of the series (6) at $s = -\epsilon$:

$$A_N(\epsilon) = \sum_{n=1}^N a_n n^\epsilon.$$

Our goal is to find for which values of ϵ it is possible to choose β_L such that the sequence $\{A_N(\epsilon)\}$ converges as $N \rightarrow \infty$.

A straightforward but quite crucial fact will be the following (mentioned previously):
If $L_1 < L_2$ then

$$\forall n_1 \in \Pi_{L_1}^\times, \forall n_2 \in \Pi_{L_2}^\times, \text{ we have } n_1 < n_2 .$$

We define

$$L^*(N) = \max\{L : \exists n \leq N \text{ with } n \in \Pi_L^\times\} .$$

By definition of $L^*(N)$, and by the fact above, if $L < L^*(N)$ then

$$\forall n \in \Pi_L^\times, \text{ we have } n \leq N$$

and therefore we can write

$$A_N(\epsilon) = \sum_{L < L^*(N)} \sum_{n \in \Pi_L^\times} a_n n^\epsilon + \sum_{n \in \Pi_{L^*(N)}^\times, n \leq N} a_n n^\epsilon .$$

The important point is that, in the first term, the inner sums range over all $n \in \Pi_L^\times$ because all these n satisfy $n \leq N$.

We would like to continue to express the inner sums in $A_N(\epsilon)$ as one-dimensional (in order to sum by parts in the desired order), so let proceed with the understanding

$$n = p_{k_{i_1}^{(1)}} \cdots p_{k_{i_M}^{(M)}} \longleftrightarrow (i_1, \dots, i_M) .$$

We have

$$\begin{aligned} A_N(\epsilon) = & \sum_{L < L^*(N)} \beta_L 2^{-XL} \sum_{n \in \Pi_L^\times} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} n^\epsilon \\ & + \beta_{L^*(N)} \alpha_{L^*(N)} \sum_{n \in \Pi_{L^*(N)}^\times, n \leq N} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} n^\epsilon . \end{aligned}$$

Define

$$(12) \quad \Omega_L(\epsilon) = \sum_{n \in \Pi_L^\times} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} n^\epsilon .$$

$$(13) \quad \Gamma(N, \epsilon) = \sum_{n \in \Pi_{L^*(N)}^\times, n \leq N} \omega_{r_2}^{i_1 i_2} \cdots \omega_{r_M}^{i_{M-1} i_M} n^\epsilon .$$

This means

$$(14) \quad A_N(\epsilon) = \sum_{L < L^*(N)} \beta_L 2^{-XL} \Omega_L(\epsilon) + \beta_{L^*(N)} \alpha_{L^*(N)} \Gamma(N, \epsilon) .$$

We aim to show that $2^{-XL} |\Omega_L(\epsilon)| \rightarrow 0$ and $2^{-XL} |\Gamma(N, \epsilon)| \rightarrow 0$, for a certain value of ϵ . This is not sufficient by itself, but along with our freedom to choose β_L it will be enough to prove the result.

We prove next that the sums (12), (13) have a large amount of cancellation. A large amount of cancellation should not be surprising, because of the $\omega^{i_1 i_2 + \dots}$ factors. Let us index the elements of Π_L^\times in increasing order:

$$\Pi_L^\times = \{n_1, \dots, n_{|\Pi_L^\times|}\} .$$

We will sum by parts:

Proposition 5 (A Version of Summation by Parts). *Suppose a_1, \dots, a_p and b_1, \dots, b_p are given. Define $B_N = \sum_{j=1}^N b_j$. Then*

$$\sum_{j=1}^p a_j b_j = \sum_{j=1}^{p-1} (a_j - a_{j+1}) B_j + a_p B_p$$

Proof. Standard. □

Applying this to $\Omega_L(\epsilon)$, $\Gamma(N, \epsilon)$, we have

$$\Omega_L(\epsilon) = \sum_{j=1}^{|\Pi_L^\times|-1} (n_j^\epsilon - n_{j+1}^\epsilon) \left[\sum_{n \in \Pi_L^\times, n \leq n_j} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_M}^{i_{M-1} i_M} \right] + n_{|\Pi_L^\times|}^\epsilon \sum_{n \in \Pi_L^\times} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_M}^{i_{M-1} i_M}$$

and, letting n^* be the maximal element of $\{n : n \in \Pi_{L^*(N)}^\times, n \leq N\}$,

$$\begin{aligned} \Gamma(N, \epsilon) &= \sum_{j: n_j \leq N} (n_j^\epsilon - n_{j+1}^\epsilon) \left[\sum_{n \in \Pi_{L^*(N)}^\times, n \leq n_j} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_M}^{i_{M-1} i_M} \right] \\ &\quad + (n^*)^\epsilon \sum_{n \in \Pi_{L^*(N)}^\times, n \leq N} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_M}^{i_{M-1} i_M} . \end{aligned}$$

Now, we need to estimate the size of a sum of the form

$$\sum_{n \in \Pi_L^\times, n \leq P} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_M}^{i_{M-1} i_M}$$

for general P , this estimate was provided by equation (11) in the preceding section (we will substitute the actual bound in place of E_L when required):

$$\left| \sum_{n \in \Pi_L^\times, n \leq P} \omega_{r_2}^{i_1 i_2} \dots \omega_{r_M}^{i_{M-1} i_M} \right| \leq E_L .$$

With this estimate, we have

$$|\Omega_L(\epsilon)| \leq 2 (\max\{n : n \in \Pi_L^\times\})^\epsilon E_L .$$

Recalling the bound (8) on the largest element of Π_L^\times , we see that, with C_M being (another) constant depending on M , and $\epsilon < 1$ assumed, we have

$$(15) \quad |\Omega_L(\epsilon)| \leq C_M 2^{\epsilon ML} L^M E_L$$

and similarly

$$(16) \quad |\Gamma(N, \epsilon)| \leq C_M 2^{\epsilon ML^*(N)} L^*(N)^M E_{L^*(N)} .$$

Using the estimate (11) on E_L , we substitute into (15), (16), and we have

$$\begin{aligned} |\Omega_L(\epsilon)| &\leq C_M 2^{\epsilon ML} L^M 2^{(\rho_1 + \dots + \rho_{M-2} + \rho_M)L} L \\ |\Gamma(N, \epsilon)| &\leq C_M 2^{\epsilon ML^*(N)} L^*(N)^M 2^{(\rho_1 + \dots + \rho_{M-2} + \rho_M)L^*(N)} L^*(N) . \end{aligned}$$

Recall that (14) tells us

$$A_N(\epsilon) = \sum_{L < L^*(N)} \beta_L 2^{-XL} \Omega_L(\epsilon) + \beta_{L^*(N)} \alpha_{L^*(N)} \Gamma(N, \epsilon) .$$

We define $\beta_L = \beta_L(\epsilon)$ to satisfy

$$\beta_L \Omega_L(\epsilon) = (-1)^{d_L} |\Omega_L(\epsilon)|$$

where $d_L = d_L(\epsilon)$ is not yet specified. This means we have

$$A_N(\epsilon) = \sum_{L < L^*(N)} (-1)^{d_L} 2^{-XL} |\Omega_L(\epsilon)| + \beta_{L^*(N)} \alpha_{L^*(N)} \Gamma(N, \epsilon) .$$

Leaving aside the $L^*(N)$ term for the moment, recall the following result from infinite series: If $a_n \geq 0$ and $a_n \rightarrow 0$, then there exists some choice of signs d_n such that the partial sums $\sum_{n \leq N} (-1)^{d_n} a_n$ converge (to some unspecified value) as $N \rightarrow \infty$. This means that, if (for some particular value of ϵ)

$$2^{-XL} |\Omega_L(\epsilon)| \rightarrow 0 \text{ as } L \rightarrow \infty$$

then there exists a choice of signs d_L such that $\sum_{L < L^*(N)} (-1)^{d_L} 2^{-XL} |\Omega_L(\epsilon)|$ converges as $N \rightarrow \infty$. Using the inequality above, we have

$$\begin{aligned} 2^{-XL} |\Omega_L(\epsilon)| &\leq 2^{-XL} L^{-(M+2)} C_M 2^{\epsilon ML} L^M 2^{(\rho_1 + \dots + \rho_{M-2} + \rho_M)L} L \\ &= C_M 2^{[(\rho_1 + \dots + \rho_{M-2} + \rho_M) - X + \epsilon M]L} L^{-1} . \end{aligned}$$

We see that, if ϵ satisfies

$$(17) \quad (\rho_1 + \dots + \rho_{M-2} + \rho_M) - X + \epsilon M = 0$$

then $2^{-XL} |\Omega_L(\epsilon)| \rightarrow 0$. Additionally, since $\alpha_{L^*(N)} \Gamma(N, \epsilon)$ is bounded by the same quantity as above (with $L^*(N)$ substituted for L), if (17) is satisfied then we also have

$$\alpha_{L^*(N)} \Gamma(N, \epsilon) \rightarrow 0$$

and therefore there will be a choice of signs $\{d_L\}$ (i.e. a choice of $\{\beta_L\}$) such that $A_N(\epsilon)$ converges as $N \rightarrow \infty$.

Let us choose

$$\epsilon = \frac{-1}{M} [(\rho_1 + \dots + \rho_{M-2} + \rho_M) - X]$$

This means (17) is satisfied, so as long as this ϵ is greater than zero, we can choose $\{\beta_L\}$ such that $f(s)$ converges at $s = -\epsilon$ on the negative real axis, and so we have proved

Proposition 6. *Let f be the Dirichlet series defined by (6) and (7). f satisfies*

$$\sigma_c \leq \frac{1}{M} ((\rho_1 + \dots + \rho_{M-2} + \rho_M) - X)$$

if the quantity on the right hand side is negative (it would not be of interest otherwise).

8. RESULTS

Examining propositions 1, 2, 4 and 6, we see that with $X = (\rho_1 + \cdots + \rho_M) \frac{M+1}{2M}$, the requirement

$$X \leq (1/2)(\rho_1 + \cdots + \rho_M + 1)$$

from proposition 1 is satisfied, and so we have a Dirichlet series f which satisfies

$$\begin{aligned}\sigma_c &\leq \frac{1}{2M^2} [(M-1)(\rho_1 + \cdots + \rho_{M-2} + \rho_M) - (M+1)\rho_{M-1}] \\ \sigma_b &\geq 0 \\ \sigma_b &\leq \frac{1}{2M} \left(1 - \frac{\rho_1 + \cdots + \rho_M}{M} \right) \\ \sigma_a &\geq \frac{M-1}{2M^2} (\rho_1 + \cdots + \rho_M)\end{aligned}$$

as long as the bound on σ_c is less than zero, i.e.

$$(M-1)(\rho_1 + \cdots + \rho_{M-2} + \rho_M) - (M+1)\rho_{M-1} < 0 .$$

Proving the results stated in the introduction, for $M = 2$ and $M = 3$, is now a matter of arithmetic:

With $M = 2$, and $\rho_1, \rho_2 = 1$, we have $\sigma_b = 0$, $\sigma_a \geq 1/4$, and $\sigma_c \leq -1/4$.

With $M = 3$, we can set $\rho_2 = \rho_3 = 1$ and then

$$\begin{aligned}\sigma_a &\geq \frac{1}{9}(\rho_1 + 2) \\ \sigma_b &\in [0, \frac{1}{18}(1 - \rho_1)] \\ \sigma_c &\leq \frac{1}{9}(\rho_1 - 1) .\end{aligned}$$

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